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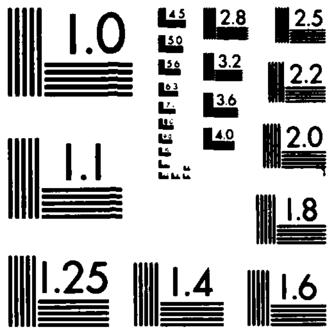


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# SIMILARITY SOLUTION FOR PERTURBATION OF A STRONG BLAST WAVE WITH RADIATION DIFFUSION IN A STRATIFIED MEDIUM

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Technical Report

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A similarity solution for the first order perturbation of a nearly spherical strong blast wave generated by a point source in a mildly stratified medium is found. The hydrodynamic equations include a radiation diffusion approximation to radiation transport which is of such a character to allow both the unperturbed and perturbed solutions to exhibit a similarity form. The cylindrically symmetric solution provides the flow in both radial and polar directions. Implications for circulation in flow fields of low-altitude atmospheric explosions are drawn.

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## TABLE OF CONTENTS

## LIST OF ILLUSTRATIONS

## SECTION 1
## INTRODUCTION

The distortion of a strong blast wave propagating in an inhomogeneous medium has been the subject of a variety of investigations. A summary of the older work is given by Zel'dovich and Raizer.[1] Kompaneets[2] looked at the problem of the propagation of an initially spherical blast wave from a point source in an atmosphere stratified in one dimension with exponentially changing density. He assumed strictly radial flow along with a hypothesis of uniform pressure in the interior of the expanding region driving a thin shock front containing all the swept up mass. This resulted in a simple analytic solution for the shock front motion. Raizer[3] found a self-similar solution for an impulsive source for planar shock propagation in an exponential medium; Grover and Hardy[4] have reported this solution along with numerical results including spherical geometry (with the source at the center of the spherical density variation). Laumbach and Probstein[5] have given an improved version of Kompaneet's calculation by avoiding any ad hoc assumption about the pressure field while retaining the assumption of radial flow. They do assume a strong blast wave and solve the hydrodynamic equations for a point source under an assumption which takes the bulk of the mass, from inside the current polar angle, to be concentrated near the shock. This is used to truncate expansions behind the shock. The result is then an approximation to both the shock location and profiles of hydrodynamic variables behind the shock which is a function of polar angle, scale height and energy.

In the limit of a homogeneous atmosphere (horizontal propagation), the solution of Laumbach and Probstein provides a good approximation to the well-known strong blast wave solution given by Taylor[6] and Sedov.[7] The Taylor-Sedov solution can be put in similarity form so that the hydrodynamic variables at any time and location can be expressed as a product of a power to time and a fixed profile which is a function of only a similarity variable which may be taken as the radius over the radius to the shock. Obviously the resulting separation of variables greatly simplifies the solution of the problem. However, as indicated by Laumbach and Probstein, the general case of an exponentially changing ambient density does not admit to a similarity solution.

The Taylor-Sedov solution is not fully useful for the realistic problem of a point energy source in a uniform atmosphere because it assumes that radiation energy and transport are insignificant. Elliott[8] has studied the circumstances under which this restriction may be lifted. He finds that when radiation transport is added to the energy conservation equation by the radiation diffusion approximation, the resulting equations sometimes still allow a similarity solution. If the temperatures are high enough for the radiant flux to affect significantly the hydrodynamics but sufficiently low that the radiation energy and pressure are small compared to the material values, a particular form for the diffusion coefficient will give a similarity solution. Remarkably enough, the form needed is quite similar to that seen experimentally. We shall exploit this fact in constructing approximate solutions for flow fields for a class of atmospheric explosions.

In this paper the flow fields influenced by radiation diffusion behind a strong shock from an energy point source will be found for an atmosphere mildly stratified in density. That is, the first order perturbation in the flow induced by a noninfinite scale height will be determined. It is found that both the *uniform* atmosphere case, done by

Elliott[8], and the first order perturbation, with an expansion coefficient inversely proportional to the scale height, allow similarity solutions for which the dependent variables can be expressed as the product of a power of time, functions of the similarity variable and simple angular functions. In fact, it can be shown that such similarity solutions exist to all orders of the expansion although this paper will be concerned only with the first order. However, each order requires a different similarity transformation.

The motivation for pursuit of this problem lies in the determination of the residual flow fields which influence the rise and distortion of fireballs from low-altitude explosions. A realistic calculation of all features of such flow requires much more than the current calculation. At very early times, explosion dynamics are dominated by radiation transport which is not diffusive and for which the radiant energy is important. Once the radiation sphere cools sufficently, a shock is launched from its surface of the radiation sphere. This strong shock propagates through the ambient atmosphere in a manner described by the current calculation. In a stratified atmosphere the propagation is distorted by a more rapid motion into lower densities. Ultimately, the shock propagation is weakened by spherical divergence so that it is no longer a strong shock. At this point the current calculation will no longer be appropriate. Ultimately the expansion of this region will cease when the pressure is reduced to ambient levels. When this pressure equilibrium fireball condition is reached, there will be a portion of the fluid through which the strong shock propagated which later is nearly isothermal due to diffusion of radiation behind the shock. It is precisely this region for which the circulation, induced by mild atmospheric stratification, which can be calculated using the methods of this paper. This region will have its circulation frozen and convected with the flow. This circulation is important in providing the characteristic torus at later times as the now underdense pressure equilibrium fireball undergoes buoyant rise. For the current calculation, only the strong shock flow fields will be found.

## SECTION 2
## RADIATION HYDRODYNAMIC EQUATIONS

### 2.1 GENERAL CASE.

The equations for inviscid hydrodynamic flow for an ideal fluid in the presence of a radiant energy flux are taken to be

$$\partial\rho/\partial t + \nabla\cdot(\rho\vec{v}) = 0 \tag{1}$$
$$\partial\vec{v}/\partial t + (\vec{v}\cdot\nabla)\vec{v} = -(\nabla p)/\rho \tag{2}$$
$$\partial p/\partial t + \gamma p\nabla\cdot\vec{v} + \vec{v}\cdot\nabla p + (\gamma-1)\nabla\cdot\vec{F} = 0 \tag{3}$$

where $\gamma$ is the adiabatic exponent and $\vec{F}$ is the radiant flux. In the radiation diffusion approximation, the flux is proportional to the temperature gradient as

$$\vec{F} = -\eta\nabla T \tag{4}$$

where $\eta$ is the diffusion coefficient which is inversely proportional to an opacity, K, given by

$$K = \frac{16\sigma T^3}{3\rho\eta} \tag{5}$$

where $\sigma$ is the Stefan-Boltzmann constant. This radiation diffusion approximation is valid when the photon mean free paths are small compared with the hydrodynamic dimensions of interest in the problem and the radiant energy and pressure are small compared with the corresponding hydrodynamic quantities. The latter conditions permit the introduction of an ideal fluid equation of state which completes the set of equations. Apart from early times, these equations should provide a reasonable description for low-altitude explosions.

The above equations appear everywhere except at the shock front. At the shock the hydrodynamic variables must meet Hugoniot jump conditions[9] which enforce mass, momentum and energy conservation through the shock. Under the conditions assumed, these will now be expressed in terms of the local ambient density $\rho_a$ and the local shock velocity $v_s$ as

$$\rho(v_s - v_\perp) = \rho_a v_s \tag{6}$$

$$v_\parallel = 0 \tag{7}$$

$$p = \rho_a v_s v_\perp \tag{8}$$

$$\frac{\gamma p}{(\gamma-1)\rho} + \frac{(v_s - v_\perp)^2}{2} - \frac{F_\perp}{\rho_a v_s} = \frac{v_s^2}{2} \tag{9}$$

where it is assumed that $F_\perp = p = v_\perp = v_\parallel = 0$ outside the shock. Here p=0 is the strong shock assumption while F=0 indicates that no radiant energy escapes the region so that total energy inside the shock will be conserved. Note that $\parallel$ and $\perp$ indicate parallel and perpendicular to the local shock surface so that the location of the shock front, including its velocity, is a part of the solution to the problem. In principle, the above equations will provide for strong shock motion generated for any ambient density $\rho_a$ and opacity once the condition of total energy conservation is imposed. However, solutions are available only for a limited set of cases where similarity conditions are met. (For the case where the strong shock condition is dropped, numerical solutions have been found by Brode[10] for the case of uniform ambient density and no diffusion.)

## 2.2 ZEROTH ORDER.

In the limit of uniform ambient density, which is the zeroth order case for an expansion in a parameter inversely proportional to the scale height, the solution for spherically symmetric strong shock problem

is known from Taylor[6] and Sedov[7] for no diffusion. It is evident from dimensional arguments that there is a similarity solution which has the properties that the shock radius R(t) is proportional to $t^{2/5}$, while the density, fluid velocity, and pressure can be expressed as a power of t (the powers are 0, -3/5, and -6/5, respectively) multiplied by profiles which are functions only of a similarity variable which can be taken as the ratio of the radius, r) to the shock radius, R(t). Relatively simple closed form solutions for the profiles exist.

When radiation diffusion is included for the constant ambient density case, it is not generally possible to find similarity solutions for the equations. However, Elliott[8] has demonstrated that for a diffusion coefficient with a suitable dependence on temperature, the similarity form can be maintained. It is easy to show that the diffusion term in the energy equation (3) will give the same separation into a power of time times a scaled profile, as do the other terms, using the same powers as indicated by Taylor provided that the diffusion coefficient, $\eta$, is proportional to 1/6th power of the temperature, T. It may be any function of the density. This means the opacity must be proportional to $T^{17/6}$ but may have any density dependence. It turns out that for temperatures below about 10 eV, the range of interest for the current problem, the Rosseland mean opacity rises sharply approximately like $T^3$ while decreasing with density. For calculational purposes, the diffusion coefficient will be taken as

$$\eta = \alpha \rho^{k} T^{1/6} \tag{10}$$

where k= -1.6 as estimated for air densities between $10^{-3}$ and $10^{-4}$ gm/cm$^3$ which gives $\alpha \approx 8.7 \times 10^{-5}$ in cgs units.

Elliott has given a technique for solving the similarity equations which involves carrying out the first integral of the energy equation analytically and making a change of variables which converts all the

unknown boundary values to appear at the (unknown) shock location. The equations are then integrated outward from the origin to find the scaled location of the shock front at which the Hugoniot conditions are met. The only drawback to this procedure is that the required scaling of the diffusion coefficient is dependent on the solution so it is necessary to first find a solution with a scaled coefficient and then determine the physical coefficient to which it corresponds. As a result it necessary to iterate to find the solution for a given physical coefficient. In practice, this is not a problem since there is a smooth relation between the scaled and physical coefficients. A part of the solution also consists of finding the constant of proportionality between the shock radius, R, and $t^{2/5}$. This is a function of E (total energy), $\rho_a$, $\gamma$, and $\eta$ and it can be determined by integrating the solution profiles for internal and kinetic energy and equating the sum to the total energy. For Taylor's case ($\eta$ = 0) it is known that for $\gamma = 1.2$ (which is a reasonable value for ionized air) the energy relation is

$$E = 11.0\ \rho_a \dot{R}^2 R^3 \tag{11}$$

and

$R = a\ t^{2/5}$ which gives a in terms of E.

For a $\rho_a$ of $5 \cdot 10^{-4}$ gm/cm$^3$ (which corresponds to approximately one scale height above sea level) and an energy of 40 kT (or $1.67 \cdot 10^{21}$ ergs) the (dimensionless) coefficient of 11.0 is reduced to 10.2 by inclusion of diffusion indicating a moderate effect near the shock which serves to increase the shock speed. (The example used is similar to that labeled as $\overline{K} = 0.5$ in Elliott's case.) Although the effects of diffusion near the shock are modest, the effects in the center of the fireball are pronounced as the temperature is forced to be fairly uniform over the bulk of the volume. This is illustrated by the plots of density, velocity, pressure and temperature shown in Figures 1-4. The plots show the profiles taken as a function of the similarity variable $x = r/R(t)$ where

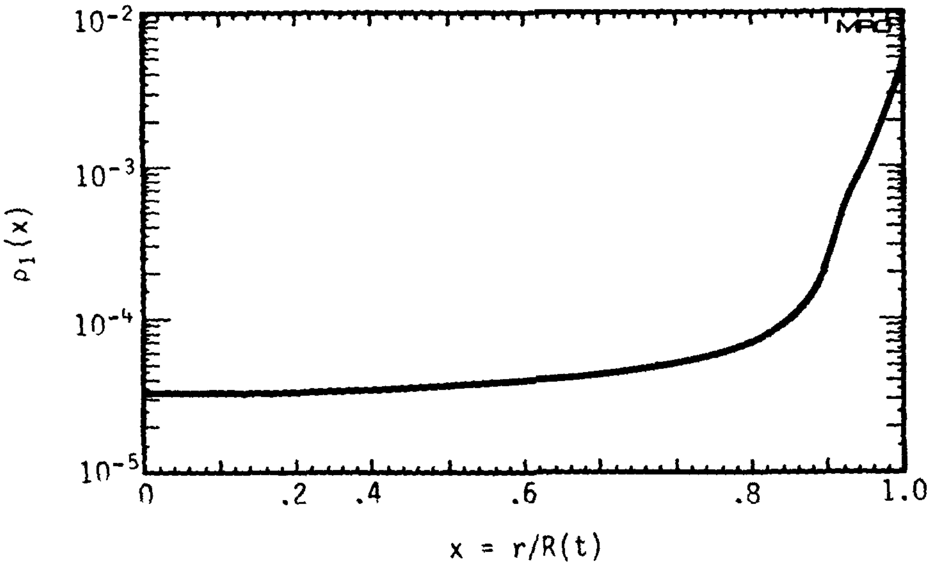


Figure 1. Density similarity profile.

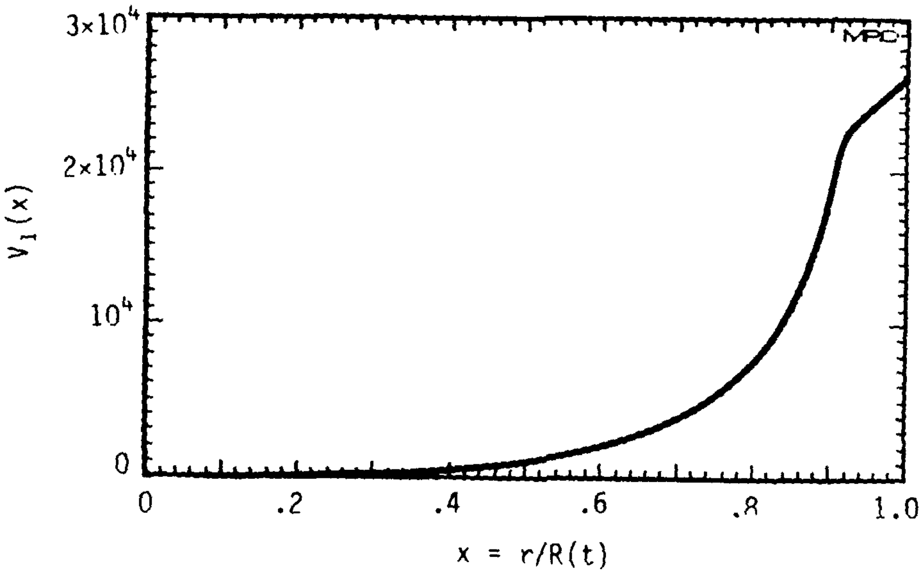


Figure 2. Velocity similarity profile.

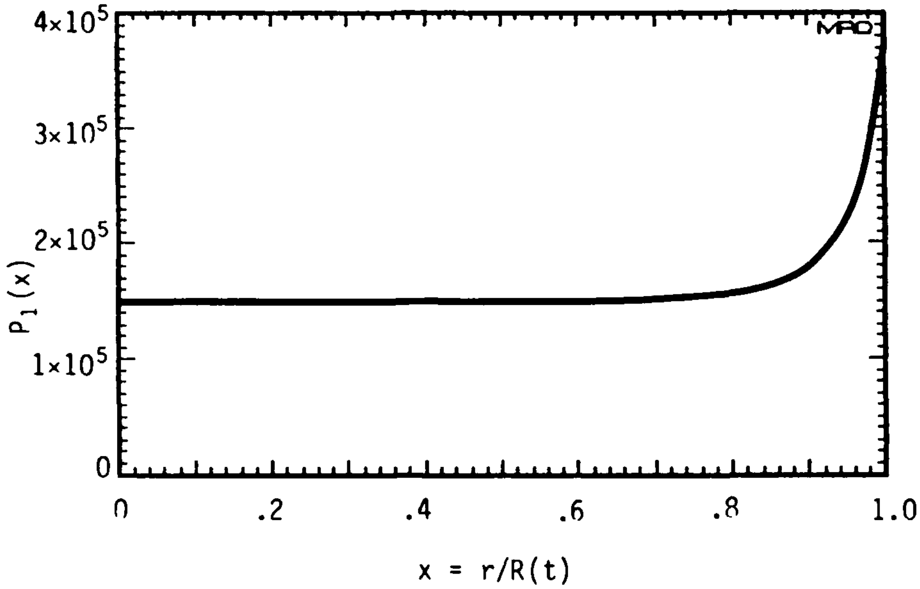


Figure 3. Pressure similarity profile.

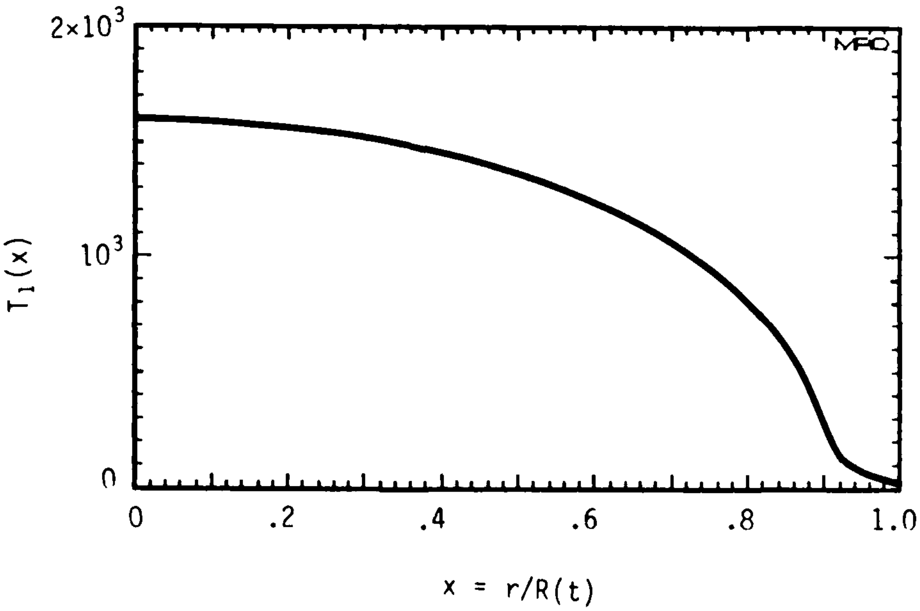


Figure 4. Temperature similarity profile.

$$\rho(r,t) = \rho_1(x) \tag{12}$$

$$v(r,t) = v_1(x)\ t^{-3/5} \tag{13}$$

$$p(r,t) = p_1(x)\ t^{-6/5} \tag{14}$$

$$T(r,t) = T_1(x)\ t^{-6/5} \tag{15}$$

where T is not independent but given by the ideal gas equation of state

$$p = \rho R^{*} T \tag{16}$$

Here $R^*$ is the gas constant given by the product of Boltzmann's constant and Avogadro's number over the mean molecular weight of the constitutents which has been taken as thirty for the numerical example.

The solutions including diffusion differ from the Taylor's case in that the temperature now is nearly uniform in the interior while the pressure and density go to nonzero values at the origin. Near the origin the pressure goes like the fourth power of x plus a constant, the density goes like the second power of x plus a constant while the velocity, to lowest order, goes like the third power of x. Near the shock, the solutions are similar to those without diffusion although the speed of the front in increased somewhat by the outward flow of radiant energy.

From the present considerations, it is important that there exists an easily obtained solution which strongly resembles that for a physical fireball which can serve a the basis for a perturbation expansion to determine the character of the effects of mild stratification of the ambient medium.

## 2.3 FIRST ORDER.

In ambient density which is not constant, the solution given above is no longer correct. However, if the density changes only slightly

over the region defined by the shock position, it is expected that this solution will be approximately correct. This fact can be exploited for the case of the atmosphere, whose density variation is almostly exclusively in the vertical direction and can be described locally as exponential with a constant characteristic scale height, H, which is imposed by an equilibrium between gravity and ambient pressure forces. The dynamical Equations 1-3 are still appropriate for this nonconstant density provided that gravitational forces are not important for development of the strong shock and associated flow. Since strong shock conditions are imposed, this is true. However, the effects of an exponential ambient density will alter the Hugoniot conditions in a mild way (so long as $R(t) \ll H$) exclusively through the altitude dependence of $\rho_a$. Since the density is not spherically symmetrical, the resulting flow will now have a polar angle dependence, including a nonradial velocity, and the possibility of vorticity where there was none before.

Previous work[1-5] indicates that when a strong shock propagates in an exponential atmosphere, the motion into decreasing density is enhanced while that in increasing density is diminished. The initially spherical pulse is thus distorted by the upward expansion which can, for a small H, result in the upwardly moving shock accelerating all the way through the top of the atmosphere (where the hydrodynamic equations are, of course, not valid). For a large, but finite, scale height, a mild distortion of the spherical pulse is expected. This distortion will now be described as a small perturbation about the constant density solution and an expansion for perturbed solution will be found.

The ambient density and its leading term is

$$\rho_a = \rho_0 \exp(-z/H) = \rho_0 \exp(-r \cos\theta/H) \approx \rho_0 (1-r \cos\theta/H) \quad (17)$$

where $\rho_0$ is the ambient density at $z=0$, $\theta$ is the polar angle and r is the spherical radius. The form of the expanded density suggests the expansion

of the perturbed solution in a Fourier series in order to define the angular dependence. When this is done, it is readily found that the series for all the dependent variables, including the boundary conditions, truncate to the following form provided that higher order terms are ignored:

$$\rho = \rho_1 + f_\rho(r,t) \cos\theta \tag{18}$$

$$p = p_1 + f_p(r,t) \cos\theta \tag{19}$$

$$v_r = v_1 + f_v(r,t) \cos\theta \tag{20}$$

$$v_\theta = f_\theta(r,t) \sin\theta \tag{21}$$

The position of the perturbed shock front is described by an auxilliary function $G(r,\theta,t)=0$ where

$$G = R(t) - r + f_R(t) \cos\theta \tag{22}$$

so $f_R$ is the perturbation in shock front location in the vertical direction. The (linearized) equations which the $f(r,t)$ functions obey are:

$$\frac{\partial f_\rho}{\partial t} + \rho \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 f_v) + \frac{2}{r} f_\theta\right) + f_\rho \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v)\right) + v \frac{\partial f_\rho}{\partial r} + f_v \frac{\partial \rho}{\partial r} = 0 \quad , \tag{23}$$

$$\frac{\partial f_v}{\partial t} + v \frac{\partial f_v}{\partial r} + f_v \frac{\partial v}{\rho r} = - \frac{1}{\rho} \frac{\partial f_p}{\partial r} + \frac{f_\rho}{\rho^2} \frac{\partial p}{\partial r} \quad , \tag{24}$$

$$\frac{\partial f_\theta}{\partial t} + v \frac{\partial f_\theta}{\partial r} + \frac{v f_\theta}{r} = \frac{f_p}{\rho r} \quad , \tag{25}$$

$$\frac{\partial f_p}{\partial t} + \gamma\, p\left(\frac{1}{r^2}(r^2 f_r) + \frac{2}{r} f_\theta\right) + \gamma f_p \left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v)\right)$$

$$+ v\frac{\partial f_p}{\partial r} + f_v \frac{\partial p}{\partial r}$$

$$+ (\gamma-1)\left(-\frac{6}{7}\alpha\left(\frac{R^*}{\gamma-1}\right)^{7/6}\right)\left\{\frac{\partial T^{7/6}}{\partial r} k \frac{\partial}{\partial r}(f_\rho \rho^{k-1}) + \frac{\partial \rho^k}{\partial r}\frac{7}{6}\frac{\partial}{\partial r}(f_T\, T^{1/6})\right. \tag{26}$$

$$+ \rho^k\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2\left(\frac{\partial}{\partial r}\left(\frac{7}{6}T^{1/6} f_T\right)\right) - \frac{2}{r^2}\frac{7}{6}T^{1/6} f_T\right]$$

$$\left. + k\, f_\rho \rho^{k-1}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(T^{7/6})\right)\right\} = 0$$

Here the subscript, one, has been dropped from the leading order terms. The boundary conditions to this order, as applied at the perturbed shock location, are

$$v\, f_R = R f_\theta \tag{27}$$

$$\rho\left(\dot{f}_R - \frac{\partial v}{\partial r} f_R - f_v\right) + (\dot{R} - v)\left(\frac{\partial \rho}{\partial r} f_R + f_\rho\right) = \rho_0\left(-\frac{\dot{R}R}{H} + \dot{f}_R\right) \tag{28}$$

$$\frac{\partial p}{\partial r} f_R + f_p = \rho_0\left(\dot{R}\left(\frac{\partial v}{\partial r} f_R + f_v\right) + v\,\dot{f}_R - \frac{R}{H}\dot{R}\, v\right) \tag{29}$$

$$\frac{\gamma R^*}{\gamma-1}\left(\frac{\partial T}{\partial r} f_R + f_T\right) + (\dot{R} - v)\left(-\frac{\partial v}{\partial r} f_R + \dot{f}_R - f_T\right)$$

$$+ \frac{\alpha}{\rho_0 \dot{R}}\left(\frac{R^*}{\gamma-1}\right)^{7/6}\left[\frac{\partial T}{\partial r} T^{1/6}\rho^{k-1} k\left(\frac{\partial \rho}{\partial r} f_R + f_\rho\right) + \frac{\partial T}{\partial r}\rho^k \frac{T^{-5/6}}{6}\left(\frac{\partial T}{\partial r} f_R + f_r\right)\right. \tag{30}$$

$$\left. + T^{1/6}\rho^k\left(\frac{\partial^2 T}{\partial r^2} f_R + \frac{\partial f_T}{\partial r}\right) + \frac{\partial T}{\partial r}\rho^k\, T^{1/6}\left(\frac{R}{H} - \frac{f_R}{\dot{R}}\right)\right] = \dot{f}_R \dot{R}$$

where the temperature perturbation $f_T$ has been used as an alternative to the pressure through the first order terms in the equation of state. It must be pointed out that higher order terms, such as $f^2$, will contain Fourier contributions which are independent of angle and which will thus couple the zeroth and second order terms. However, when only first order terms are kept, the only terms constant in angle are those from the zeroth order.

The perturbation equations for four f functions consist of a linear set of four coupled homogeneous second order partial differential equations in two dependent variables subject to four linear but inhomogeneous boundary conditions applied at a position fixed by $f_R$ which is to be determined. The task of solving would be formidable except for one fact. It is possible to find a similarity solution to the perturbed equations in a manner completely analogous to that seen at zeroth order! It is a straightforward matter to verify that when the f's are written as

$$f_\rho(r,t) = t^{2/5}\, g_\rho(x) \tag{31}$$

$$f_p(r,t) = t^{-4/5}\, g_p(x) \tag{32}$$

$$f_v(r,t) = t^{-1/5}\, g_v(x) \tag{33}$$

$$f_\theta(r,t) = t^{-1/5}\, g_\theta(x) \tag{34}$$

and

$$f_R(t) = t^{4/5}\, g_R \tag{35}$$

where $x = r/R(t)$ as before, the equations are reduced to similarity form in the variable x. Consequently, the problem is reduced to solving a set of coupled second order linear differential equations in one dependent variable. The equations are

$$\begin{aligned}\left(\frac{v}{a} - \frac{2x}{5}\right) g'_\rho + \frac{\rho}{a}\, g'_v = &- \frac{2}{5} g_\rho - \rho\left(\frac{2}{ax} g_v + \frac{2g_\theta}{ax}\right) \\ &- g_\rho\left(\frac{2}{ax} v + \frac{v'}{a}\right) - g_v(\rho'/a)\end{aligned} \tag{36}$$

$$g_\theta'\left(\frac{v}{a} - \frac{2x}{5}\right) = \frac{g_p}{ax\rho} - g_\theta\left(\frac{v}{ax} - \frac{1}{5}\right) \tag{37}$$

$$\begin{aligned}\left(\frac{v}{a} - \frac{2}{5}x\right)g_v' + \frac{p}{a\rho^2}\, g_\rho' + \frac{R^*}{a}\, g_T' &= - g_v\left(\frac{v'}{a} - \frac{1}{5}\right) + \frac{g_\rho}{\rho^2}\frac{p'}{a} \\ &\quad - \frac{1}{a\rho}\left[p'\left(\frac{g_p}{p}\right) - p\left(\rho'\,\frac{g_\rho}{\rho^2} + T'\,\frac{g_T}{T^2}\right)\right]\end{aligned} \tag{38}$$

$$\begin{aligned}&R^*[\rho(-\tfrac{4}{5}\, g_T - \tfrac{2}{5}\, x g_T') - g_T(\tfrac{2}{5}\, x\rho') + T(\tfrac{2}{5}\, g_\rho - \tfrac{2}{5}\, x g_\rho') + g_\rho(-\tfrac{6}{5}\, T - \tfrac{2}{5}\, xT')] \\
&\quad + \gamma p\,(g_v'/a + \tfrac{2}{ax}(g_v + g_\theta)) + \gamma R^*(\rho\, g_T + T\, g_\rho)(\tfrac{2}{ax}\, v + v'/a) \\
&\quad + v\, R^*(\rho\, g_T'/a + g_T\rho'/a + T\, g_\rho'/a + g_\rho\, T'/a) + g_v p'/a \\
&\quad + (\gamma-1)(-\tfrac{6}{7}\,\alpha(\tfrac{R^*}{\gamma-1})^{7/6})\} \\
&\quad \tfrac{7}{6a}\, T^{1/6}\, T'(k(k-1)\rho^{k-2} g_\rho \rho'/a + k\rho^{k-1} g_\rho'/a) \\
&\quad + k\rho^{k-1}(\rho'/a)(\tfrac{7}{36}\, g_T\, T^{-5/6}\, T'/a + \tfrac{7}{6}\, T^{1/6}\, g_T'/a) \\
&\quad + k g_\rho\, \rho^{k-1}(\tfrac{14}{6ax}\, T^{1/6}\, T'/a + \tfrac{7}{36}\, T^{-5/6}(T'/a)^2 + \tfrac{7}{6a^2}\, T^{1/6}\, T'') \\
&\quad + \tfrac{2\rho^k}{ax}\, (\tfrac{7}{36}\, g_T\, T^{-5/6}(T'/a) + \tfrac{7}{6a}\, T^{1/6}\, g_T') \\
&\quad + \tfrac{7\rho^k}{36}\, (g_T'\, T^{-5/6}\, T'/a^2 - \tfrac{5}{6}\, T^{-11/6}\, g_T(T'/a)^2 + g_T\, T^{-5/6}(T''/a^2)) \\
&\quad + \tfrac{7\rho^k}{6}\, (\tfrac{1}{6a^2}\, T^{-5/6} T'\, g_T' + T^{1/6}\, g_T''/a^2 - \tfrac{14\rho^k}{6a^2x^2}\, T^{1/6}\, g_T\} = 0\end{aligned} \tag{39}$$

where the equation of state at this order is

$$\frac{g_T}{T} = \frac{g_p}{p} - \frac{g_\rho}{\rho} \quad . \tag{40}$$

The Hugoniot conditions applied at x=1 are

$$v\, g_R = a g_\theta \tag{41}$$

$$\rho\left(\left(\frac{4}{5} - \frac{v'}{a}\right)g_R - g_v\right) + \left(\frac{2a}{5} - v\right)\left(\rho' g_R/a + g_\rho\right) = \rho_0\left(\frac{4}{5}\, g_R - \frac{2}{5}\frac{a^2}{H}\right) \tag{42}$$

$$p' g_R/a + g_p = \rho_0\left(\frac{2a}{5}\left(\frac{v'}{a}\, g_R + g_v\right) + \frac{4}{5}\, v\, g_R - \frac{2}{5}\frac{a^2}{H}\, v\right) \tag{43}$$

$$\begin{aligned} &\frac{\gamma R^*}{\gamma-1}\left(T' g_R/a + g_T\right) + \left(\frac{4}{5}\, g_R - g_v - v' g_R/a\right)\left(\frac{2}{5}\, a - v\right) \\ &+ \frac{5\alpha}{2\rho_0 a}\left(\frac{R^*}{\gamma-1}\right)^{7/6}\left[\frac{T'}{a}\, T^{1/6} \rho^{k-1} k\left(\rho_1'\, g_R/a + g_\rho\right) + \frac{T'}{6a}\, \rho^k\, T^{-5/6}\left(T' g_R/a + g_T\right)\right. \\ &\left. + \frac{T'}{a}\, \rho^k\, T^{1/6}\left(\frac{a}{H} - \frac{2g_R}{a}\right) + T^{1/6}\rho^k\left(T'\, g_R/a^2 + g_T'/a\right)\right] = \frac{8}{25}\, a\, g_R \end{aligned} \tag{44}$$

In the presence of the perturbation, the total energy must be unchanged. This condition is automatically imposed at this order by the symmetry of the angular dependence. The total momentum must also be constant (and equal to zero) in the presence of the perturbation. This condition is not automatically satisfied and it serves as an additional condition. The vertical momentum integral is

$$P_Z \propto g_R v(1)\, \rho(1) + a \int_0^1 x^2 dx\, \left[g_\rho v + \rho g_v - 2\rho g_\theta\right] \tag{45}$$

The second order energy equation can be trivially changed into two first order equations if the dependent variable

$$f_q = df_T/dx \tag{46}$$

is introduced and the second derivative of $f_T$ is replaced by $df_q/dx$. There are then five first order equations with two unknown parameters which fix the values at x=1. These are $f_q(1)$ and $g_R$. Note that $g_R$ along

with the zeroth order solutions fully fix the Hugoniot conditions at this order. In principle, there can be a family of solutions in these two parameters for which only parameter can be specified independently and still allow the momentum to vanish. Consequently, one more condition can be imposed on physical grounds. The required condition is that the perturbation in the temperature, $f_T$, vanish at the origin. If it does not, there will be a discontinuity in temperature there in crossing the horizontal plane of the source since the coefficient of $f_T$ is $\cos\theta$ which changes sign across the plane. Now it is possible, in principle, to solve the equations by exploring the solutions in the parameters $f_q(1)$ and $g_R$ and selecting that which causes the total momentum and $f_T(0)$ to vanish.

## SECTION 3
## SOLUTIONS TO SIMILARITY EQUATIONS

The set of five coupled linear first order differential equations, which contain the known zeroth order functions as variable coefficients, can be integrated numerically through standard techniques. However, the presence of the diffusion term makes integration from the shock front to the origin very unstable. The term has a corresponding stabilizing effect for outward integration. Unfortunately, the natural specification of boundary conditions at $x=1$ calls for inward integration. In principle, it is possible to solve the equations by choosing parametric inner starting conditions and searching for that set of parameters which lead to matching of the boundary conditions upon outward integration. In practice, the coupled nature of the boundary conditions in conjunction with numerical inaccuracy makes this very difficult because a nearly singular set of conditions must be met.

Since the five coupled equations are linear and homogeneous, it is possible, at least in principle, to find a set of five independent solutions which can be summed with with arbitrary coefficents to form the general solution. The boundary conditions are linear in the dependent variables so, given the general solution, the coefficients can be solved for using a simple system of linear equations. The only problem is thus in finding a set of independent solutions.

Independent solutions will give different small x behavior. The character of the small x behavior of the zeroth order solutions is known; this can be used to determine possible behavior of the perturbed solutions. Once this is determined, numerical continuations out to the

boundary and appropriate combinations to fit the Hugoniot conditions along with momentum conservation can be found. The zeroth order solutions have the following form:

$$
\begin{aligned}
v_1 &\approx v_x x^3 + \ldots \\
\rho_1 &\approx \rho_x(1 + \mu x^2 + \ldots) \\
p_1 &\approx p_x(1 + \nu x^4 + \ldots) \\
T_1 &\approx T_x(1 - \mu x^2 + \ldots)
\end{aligned}
\tag{47}
$$

where the sub-x quantities along with $\mu$ and $\nu$ are known constants. Using the leading order terms along with the equations of motion, a set of five arbitrary coefficients, A, B, C, D and E, define the small x behavior is available from independent solutions to the perturbed equations:

$$
\begin{aligned}
g_v &= A\,x^{-1/2} + B + E\,x^{-3} \\
g_\psi &= -3/4\,A\,x^{-1/2} - B + 1/2\,E\,x^{-3} \\
g_T &= C\,x + D\,x^{-2} + E\tau \\
g_p &= a\,\rho_x/5p_x\,B\,x + a\rho_x/2\,E\,x^{-2} \\
g_\rho &= \rho_x(a\,\rho_x/5p_x\,B - C/T_x)\,x + \rho_x(-D/T_x + a\,\rho_x/2p_x E)x^2 \\
g_q &= C - 2\,D\,x^{-3}
\end{aligned}
\tag{48}
$$

where $\tau$ is a nonzero constant which can be expressed in terms of the xeroth order parameters. Of course, each of the five independent solutions, A-E, will produce some nonzero behavior for each of the dependent variables. However, only the dominant or divergent terms are exhibited above. The five independent solutions have been found numerically by integration outward from some small value of x using the small x behavior from each in turn as the initial condition. In some cases it is necessary to specify the leading order term for each of the five dependent variables in order to assure getting independent solutions.

There are six constants, A-E plus $g_R$, available which are to be determined with four boundary conditions at x=1, a net zero momentum and the requirement of a continuous temperature at the origin. The last condition requires that there be no contribution from the D solution. It is trivial to determine the constants from the resulting linear conditions and the solutions are shown in Figures 5-8 for the particular constants indicated in the previous section. The solution has no contribution from the E component and gives $g_R = 0.69\ a^2/4H$. This means that the perturbed temperature vanishes at the origin. Since $g_T$ is multiplied by $\cos\theta$ to form the temperature, $g_T$ must vanish at x=0, otherwise there would be a temperature discontinuity along the axis through the origin. Such a discontinuity would be unphysical since the diffusion term must smooth out this behavior.

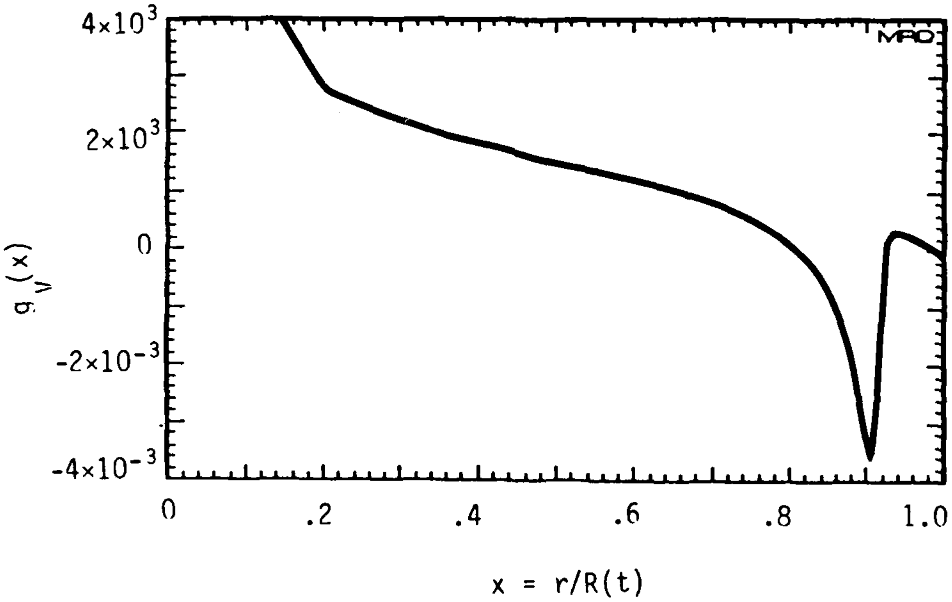


Figure 5. Perturbed radial velocity similarity profile.

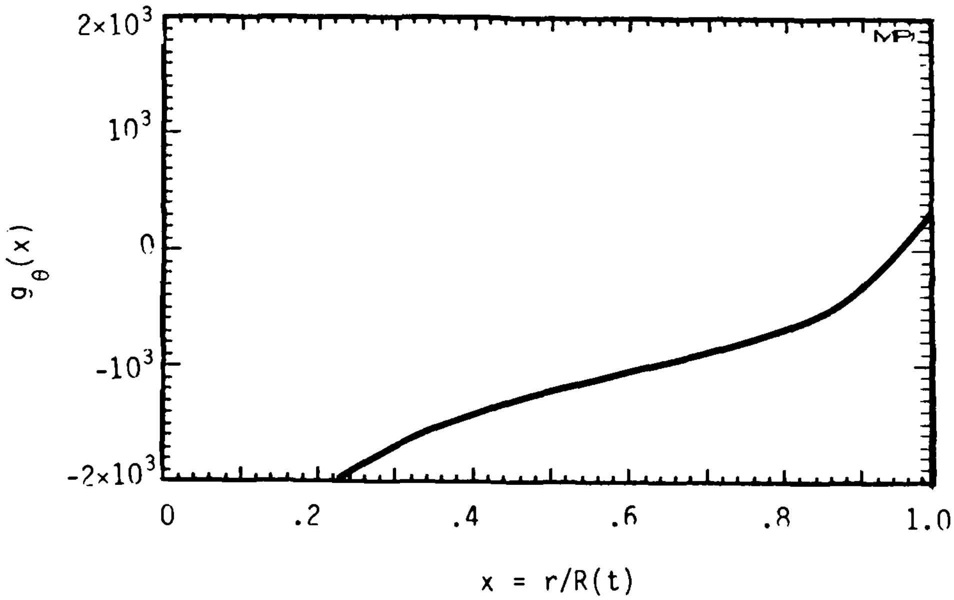


Figure 6. Perturbed polar velocity similarity profile.

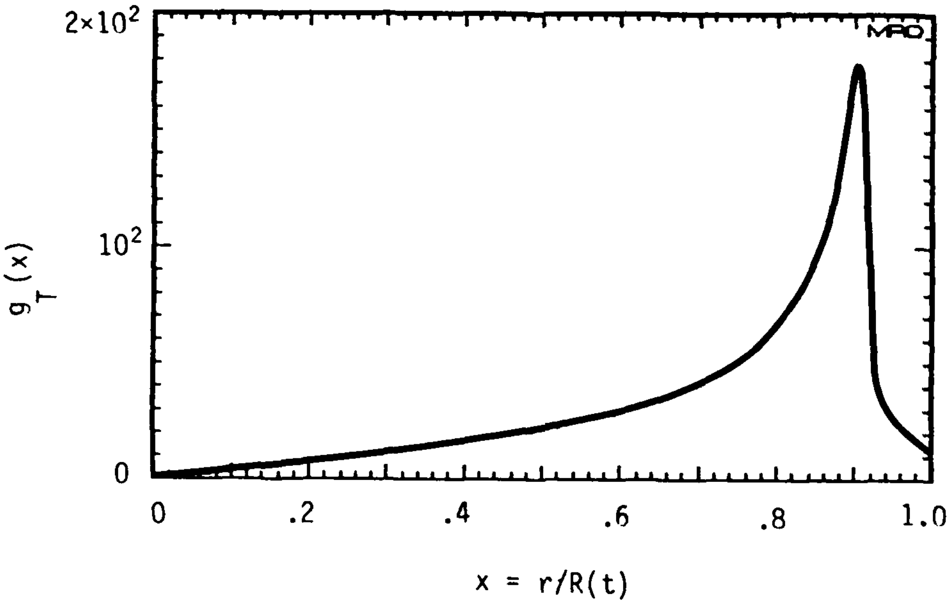


Figure 7. Perturbed temperature similarity profile.

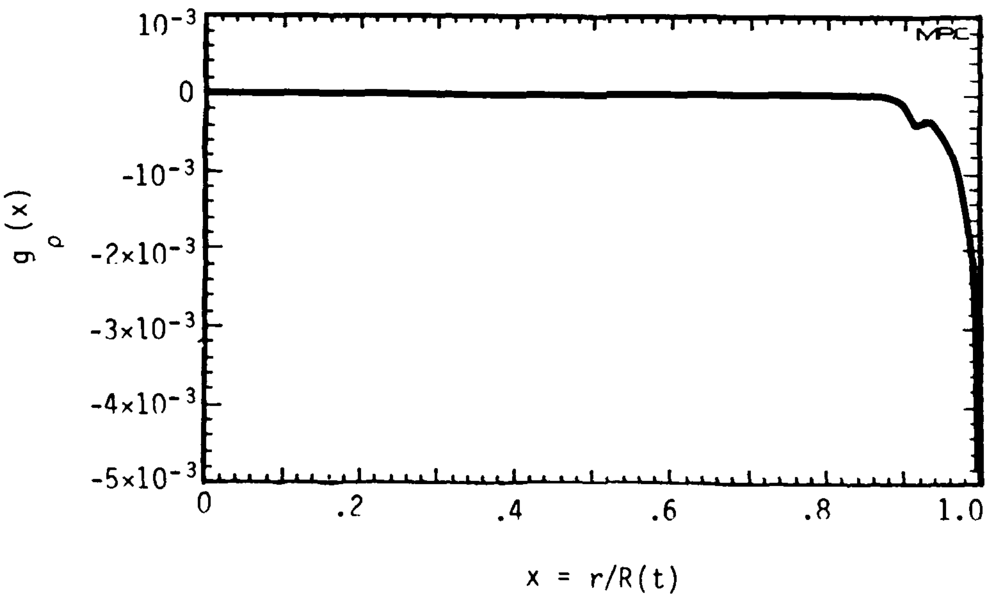


Figure 8. Perturbed density similarity profile.

## SECTION 4
## VORTICITY AND VELOCITY AT PRESSURE EQUILIBRIUM

The vorticity generated in the interior of the early-time fireball by the passage of the blast wave can result in late-time flow fields which may be important for subsequent dynamics. Aspects of this behavior have been given in detail in our previous report[11]. We now briefly review the key features of that report.

The vorticity in the flow field comes exclusively from the exponential perturbation of the ambient atmosphere, that is the first order flow calculated in the previous section. This vorticity is

$$\begin{aligned}\vec{\omega} = \nabla \times \vec{v} &= (f_\theta/r + f_v/r + \partial f_r/\partial r)\sin\theta\, \hat{e}_\phi \\ &= t^{3/5}\, \omega(x)\, \sin\theta\, \hat{e}_\phi\end{aligned} \tag{49}$$

where $\omega(x)$ contains the residual scaled dependence on position. At small ranges, $x \ll 1$, we note that the zeroth order temperature is nearly constant according to equation (47). Since the convection of the vorticity with the flow is generally described by

$$(\partial/\partial t + \vec{v}\cdot\nabla)\, \vec{\omega}/\rho = \vec{\omega}/\rho \cdot \nabla\vec{v} + \nabla T \times \nabla S \quad , \tag{50}$$

where S is entropy, a uniform temperature, T, causes the last term to vanish. Consequently, noting that $\vec{v}$ is in the radial direction and $\vec{\omega}$ is in the $\phi$ direction, the equation reduces to

$$\partial/\partial t(t^{-3/5}\omega/\rho) = v/\rho(\omega/r - \partial\omega/\partial r)^{-3/5} \tag{51}$$

The behavior of v and $\rho$ forces

$$\omega(x) = \omega_0 x^{-3/2} \tag{52}$$

where $\omega_0$ is a constant, if the vorticity convection is to be satisfied for no temperature gradient. The small x behavior of $\omega$ is, in fact, exactly this based on the use of equation (47). Equation (51) thus means $\vec{\omega}$ can be written as a function of r alone, independent of t. This means that, at small x, the vorticity is constant at a fixed position given by r and $\theta$ during the strong blast wave phase of the motion.

Following the strong blast phase, a residual fireball remains which is nearly isothermal. The fireball expands until it is nearly in pressure equilibrium with the surrounding air. At this point the initial flow fields are greatly altered through the doing of work on expansion; however, there will be residual flow associated with the blast wave generated vorticity imposed by the initial gradient in the ambient atmosphere. It is this pressure equilibrium flow field which we wish to estimate.

The altered vorticity distribution resulting from the nearly uniform expansion of the isothermal fireball can be found from equation (50) by taking $\nabla T=0$ and using $\vec{\omega}$ from equation (49) as the initial vorticity. The pre-pressure equilibrium velocity field now must be that which is dictated by fireball expansion; we ignore the contribution of the perturbed flow. Under this condition the vorticity equation can be written as

$$D/Dt(\vec{\omega}/\rho/r) = 0 \tag{53}$$

so that the final $\omega$ distribution can be expressed in terms of the uniform expansion and the intial distribution. As indicated in Reference 11 the final vorticity at pressure equilibrium becomes

$$\vec{\omega}_f = \omega_0(\rho_f/\rho_i)^{1/6}(a/r)^{3/2}\sin\theta\,\hat{e}_\phi \tag{54}$$

where the density ratio, $\rho_f/\rho_i$, is that of the final expanded fireball to that before expansion but after the blast wave has set up the vorticity.

The generation of velocities from the final vorticity field can be accomplished by noting that once pressure equilibrium has been reached, the subsequent flow is nearly incompressible. Consequently, the residual rotational velocity field can be written in terms of a vector potential, $\vec{A}$, where

$$\vec{v} = \nabla \times \vec{A} \tag{55}$$

and $\vec{A}$ is given in terms of the vorticity as

$$\vec{A} = (4\pi)^{-1} \int d^3r' \; \vec{\omega}\,(r')/\left|r-r'\right| \tag{56}$$

so that given $\vec{\omega}$, $\vec{v}$ can be determined.

The estimated velocity fields as well as the vorticity are smaller than suggested by the results in Reference 11 by a factor of about ten; the previous calculation overestimated $\omega_0$ by this amount. However this just scales the results so the structure seen before was correct. The calculation of Reference 11 differed from the current one in that it assumed a zeroth order temperature profile rather than including the radiation diffusion term. Examples of the pressure distribution velocity profiles found using the radiation diffusion term for the 40 kT example at low altitude are shown in Figures 9-11 for three different polar angles. The central vertical velocity is about 8 meters/second.

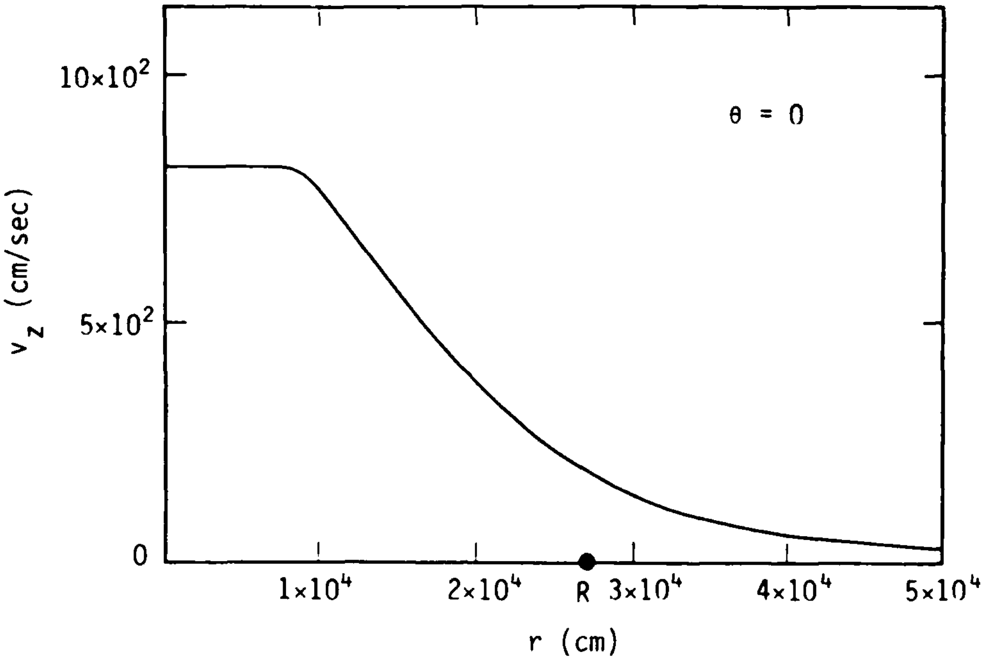


Figure 9. Vertical velocity versus radius at pressure equilibrium time at $\theta = 0$ (vertical direction).

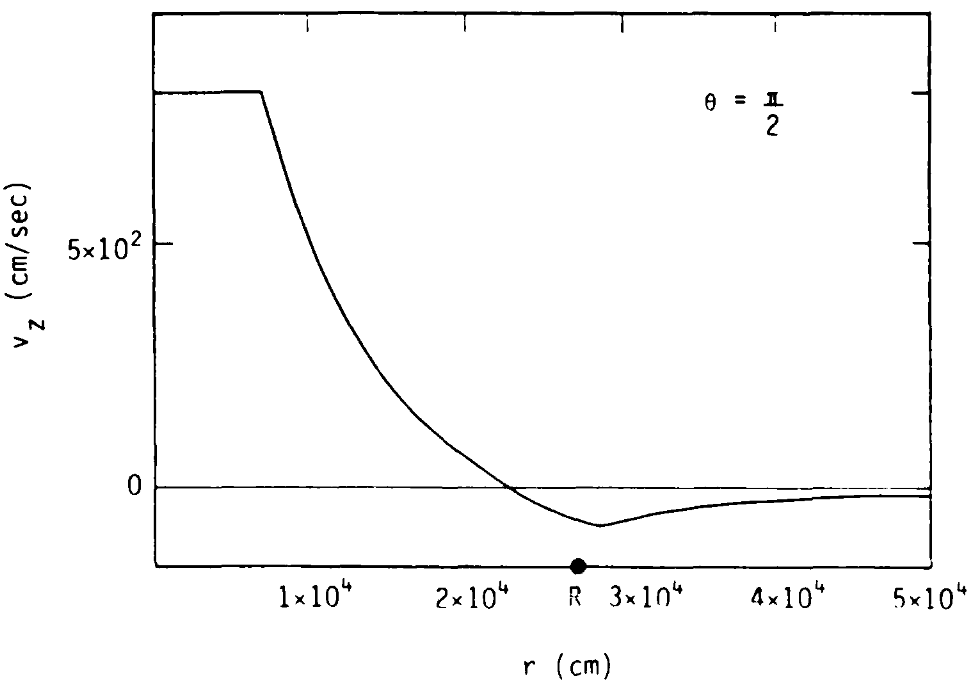


Figure 10. Vertical velocity versus radius at pressure equilibrium time at $\theta = \pi/2$ (horizontal direction).

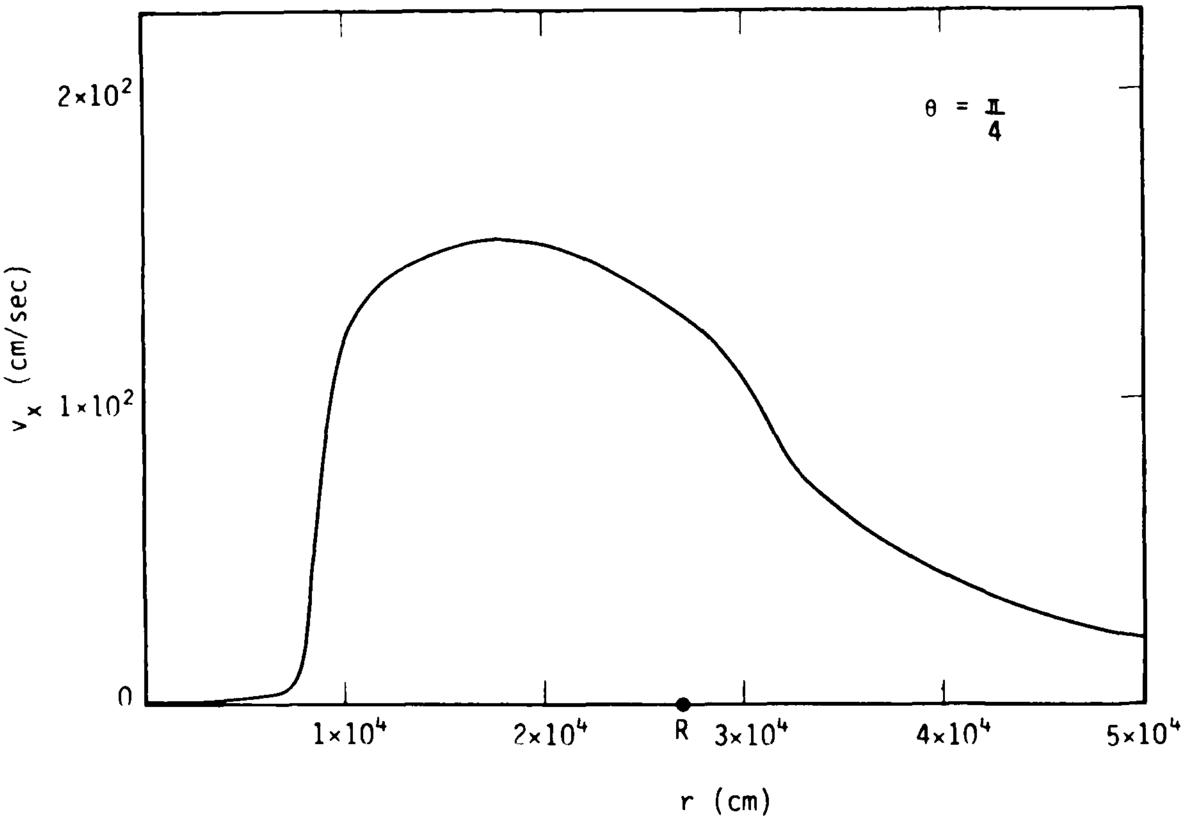


Figure 11. Vertical velocity versus radius at pressure equilibrium time at $\theta = \pi/4$ (45° off vertical direction).

## SECTION 5
## DISCUSSION

A similarity solution for the first order perturbation of a nearly spherical strong blast wave, generated by a point source in a mildly stratified medium, has been found. The hydrodynamic equations, in two spatial dimensions, include a radiation diffusion approximation to radiation transport such that it is appropriate for the problem of low altitude atmospheric explosions. The form for radiation diffusion is of such a character to allow both the unperturbed and perturbed solutions to exhibit a similarity form. The resulting cylindrically symmetric numerical solution provides the flow in both radial and polar directions as well as a description of the distortion of the nonspherical shock front. It is determined that the stratified atmosphere allows an increase in shock speed in the direction of decreasing density and to lowest order, the additive correction factor can be written as a constant coefficent times $a^2/4H \times t^{4/5} \times \cos\theta$ where $\theta$ is the polar angle and t is the time. The simplified calculations by Kompaneets gave a constant coefficent of one while the better calculation of Laumbach and Probstein, which still assumed radial flow, gave the constant as 0.75. The current calculation gives 0.69 indicating that the inclusion of nonradial flow tends to reduce the distortion of the shock front.

The perturbed flow fields have nonzero vorticity, in contrast to the radial flow case. This vorticity is generated by the passage of the blast wave through the nonuniform atmosphere. Since the hot interior of the fireball tends to have a rather uniform temperature, any circulation generated before the material passes inside will remain and be convected along with the larger zeroth order flow field. This can provide a means

for determining the flow fields in some of the material at times after the strong shock assumption or the perturbation expansion cease to be valid. The residual vorticity may be the dominant flow at times the order of one second after the fireball has ceased expanding and reached an approximate pressure equilibrium. The vorticity then may play a significant role in the subsequent rise and torusing of the fireball.

The use of a perturbation expansion limits the value of the current result to cases and times for which the shock radius is small compared to a scale height. However, the application of a similarity solution greatly reduces the work required. It might be wondered if the result here for a perturbation similarity solution can be extended so as to permit the calculation of higher order terms to expand the set of cases which can be studied. The answer is yes and it can be demonstrated that all orders in the perturbation expansion can be expressed in similarity form so that the solution is the product of a power of time multiplied by a function of the same similarity variable along with a further function of polar angle. The resulting problem is tractable but it is not as nice as the first order case in that the angular terms now contain contributions to lower Fourier orders. For example, the second order terms supply terms independent of angle as well as terms proportional to $\cos(2\theta)$. Furthermore, the perturbation equations become even more ponderous with increasing order as one would expect. The appendix provides details of the second order perturbation equations.

## SECTION 6
## LIST OF REFERENCES


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## APPENDIX

## EXTENSION OF SIMILARITY PERTURBATION TECHNIQUE

### A.1 GENERAL FORM.

As illustrated in the main body of the note, it is possible to solve the first order pertubation problem of a strong blast wave in a mildly stratified atmosphere by finding the zeroth order solution, which is known to permit a similarity solution, and using it as the basis for a perturbation to the lowest order in the stratification parameter. The remarkable feature of these first order equations is that they also admit a similarity solution in the same similarity variable as used for zeroth order. In this appendix we shall demonstrate that this result continues to hold for all orders of the perturbation expansion and the character of the required functions are provided at arbitrary order; the resulting equations for the second order perturbation will explicitly written down. For this purpose we shall ignore the complications associated with the radiation diffusion term. However, the same line of reasoning holds in that case as well if the form of the opacity is chosen so as to permit the similarity solution at zeroth order.

The hydrodynamic equations outside the shock are

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p \qquad (57)$$

$$\frac{\partial p}{\partial t} + \gamma\rho\nabla\cdot\vec{v} + \vec{v}\cdot\vec{\nabla}p = 0$$

where viscosity is ignored and an ideal gas equation of state

$$p = \rho R^{*} T \tag{58}$$

is assumed.

For a strong shock the boundary conditions which must be met at the (unknown) location of the shock front are

$$\begin{aligned}
\frac{\rho(s)}{\rho_a} &= \frac{\gamma+1}{\gamma-1} \\
v_{\perp}(s) &= \left(\frac{2p(s)}{(\gamma+1)\rho_a}\right)^{1/2} \\
v_{\parallel}(s) &= 0 \\
\frac{\partial G}{\partial t}\,\frac{1}{|\nabla G|} &= \left(\frac{(\gamma+1)p(s)}{2\rho_a}\right)^{1/2}
\end{aligned} \tag{59}$$

where $G(r,t) = 0$ defines the shock location, s, and $\rho_a$ is the ambient density.

The parallel and perpendicular directions are relative to the local shock front.

For a uniform atmosphere with zero external ambient pressure (which defines the strong shock condition) the solution must be expressable in similarity form since there are no length scales associated with the problem. It can easily be shown that the solution can be written as

$$\begin{aligned}
p &= p_0(x)t^{-6/5} \\
\rho &= \rho_0(x) \\
v_r &= v_0(x)t^{-3/5}
\end{aligned} \tag{60}$$

and

$$G = at^{2/5} - r$$

where a is dictated by the total energy in the problem.

The similarity variable x is the ratio of the radius, r, to the shock radius, $at^{2/5}$, and the time dependence is separable. By symmetry, there is no angular dependence.

Consider the problem imposed by a pertubation in the ambient density so that it is no longer uniform but is a function of only one cartesian dimension. For a realistic atmosphere, the dependence is usually taken as

$$\rho_a = \rho_b \exp(-z/H) \tag{61}$$

where the scale height H completely characterizes the variation. (For the following discussion, any mild analytic variation of density which can be expressed in terms of a single length parameter is equivalent: the result is not unique to exponential density variation.) The scale height enters the problem exclusively through the appearence of ambient density in boundary conditions.

The expansion of the ambient density is

$$\rho_a = \rho_b\left(1 - \frac{r\cos\theta}{H} + \frac{1}{2}\left(\frac{r\cos\theta}{H}\right)^2 - \ldots\right) \tag{62}$$

where the natural expansion parameter which appears is r/H and so the usual small perturbation restriction suggests that the truncated results can be expected to valid only if the radius of the shock is much less than the scale height.

The angular dependence of the perturbation terms in the density are simply factorable powers of $\cos(\theta)$. Consequently, since the perturbation expansion hydrodynamic equations, and boundary conditions, involve at most multiplication and differentiation, it will always be possible to express the solutions any order in terms of a truncated Fourier series; the number of Fourier terms must increase with increasing perturbation order.

Consider a candidate series for the perturbation series which has the similarity form with individual terms showing separable angular and time dependence:

$$\rho = \rho_0(x) + \rho_1(x,\theta)t^{\rho 1} + \rho_2(x,\theta)t^{\rho 2}$$

$$p = p_0(x)t^{-6/5} + p_1(x,\theta)t^{p1} + p_2(x,\theta)t^{p2} + \dots$$

$$v_r = v_0(x)t^{-3/5} + v_1(x,\theta)t^{v1} + v_2(x,\theta)t^{v2} + \dots \tag{63}$$

$$v_\theta = \Theta_1(x,\theta)t^{\theta 1} + \Theta_2(x,\theta)t^{\theta 2} + \dots$$

and an auxiliary function, G, giving the position of the shock front

$$G = at^{2/5} + G_1(\theta)t^{G1} + G_2(\theta)t^{G2} + \dots -r \tag{64}$$

where r is the shock position at polar angle $\theta$ for $G(r,\theta,t)=0$. (Note that $\nabla G$ defines the normal to the shock front.) The similarity variable is $x = r/R(t)$ where $R(t)=at^{2/5}$ is the zeroth order shock position. We now ask if such a similarity form can provide a solution to the equations together with boundary conditions.

Consider that the proposed solution is inserted into the full equations and the resulting equations are expanded and written as a sum of terms, each being a particular order in the perturbation series, each as coefficent of powers of the expansion parameter, 1/H. If our proposed similarity condition is to be met, the individual orders must have the same factorable time dependence in each of its contributions from all appropriate orders of the dependent variables. However, the different orders may not, in fact, cannot, have the same time dependence.

In order for this to be true certain restrictions are required based on the character of the original equations. For example, the continuity equation will require that the $n^{th}$ order density contribution less one power of time must have the time power as the product of the $i^{th}$ order in density and the $(n-i)^{th}$ order of velocity aside from the effects of the divergence operation. It is easily seen that the divergence operation can be expressed as an operation in the x and $\theta$ variables along with a division by R(t) which is independent of order. Consequently, since the i=n case is proper (-1, for $\partial/\partial t$ = -3/5, for $v_0$ plus -2/5, for 1/R(t) -- all of which is independent of the $n^{th}$ order density power), it remains only to assure that this holds for $n \neq i$. It is evident that this requires that the increments in the powers of time for successive orders of both density and velocity must be equal and equal for all orders. Inspection of the other hydrodynamic equations indicates this must be true for all the dependent variables. Note that this behavior forces the product of an arbitrary number of such series to hold the same property - that of uniform time power increments between orders.

Note that we have not yet imposed any conditions concerning the boundary behavior, which must include ambient density variation. We have only determined an allowable form for a perturbation expansion which meets the similarity condition. It remains to be seen that this is consistent with the perturbed density and boundary conditions. The expansion of the ambient density can be written as

$$\rho = \rho_b(1 - x\ at^{2/5}\cos\theta/H + (x\ at^{2/5}\cos\theta/H)^2/2 - \ldots) \quad (65)$$

which suggests that the desired power of time increment must be 2/5. However, it must be remembered that the shock conditions are imposed at the actual shock front whose perturbed location (not x=1) is to be determined as a part of the solution. Generally the evaluation of an nth order hydrodynamic variable, say the pressure, at the shock will have the form:

$$p_n + \sum_i f_i t^{fi} \frac{\partial p_{n-i}}{\partial r} + \sum_k \sum_j f_k t^{fk} f_j t^{fj} \frac{\partial^2 p_{n-k-j}}{\partial r^2} + \sum \sum \sum \ldots \quad (66)$$

all evaluated at x=1. Since the $\partial/\partial r$ supplies a extra $t^{-2/5}$ dependence, all terms at $n^{th}$ order will have the same power of time if the increment in the powers of the expansion of G is 2/5 just as for the ambient density.

Consequently, the variables evaluated at the shock front all have the desired behavior of associating a single power of time with a particular order while incrementing that power uniformly by 2/5 with each order. Since the leading order has the appropriate power for all of the boundary conditions, all orders must also. Therefore the postulated form for perturbation similarity solutions exists for the problem at hand. The dependence of the dependent variables at arbitrary $n^{th}$ order can now be written as

$$\begin{aligned}
\rho &\sim \rho_n(x,\theta)t^{2n/5} \\
p &\sim p_n(x,\theta)t^{2n/5-6/5} \\
v &\sim v_n(x,\theta)t^{2n/5-3/5} \\
G &\sim G_n(\theta)t^{2n/5+2/5}
\end{aligned} \quad (67)$$

In fact, as indicated before, the angular dependence for each term is separable in the above expressions and the $n^{th}$ order dependence can be expressed as a truncated Fourier series terminating at $\cos(n\theta)$ or $\sin(n\theta)$ depending on the parity of the variable.

## A.2 SPECIFIC EQUATIONS AT 2nd ORDER.

The form and solution for the first order perturbation terms have been given in the main text for a specific case. At higher orders the expansions becomes increasingly complex and it is doubtful if there is utility in forming the general term. However, it is illustrative to go one step further and explicitly provide the second order equations since this brings out the essential features of the angular dependence. To be consistent with the main text the expansion is taken to be

$$\rho(\vec{r},t) = \rho_0(x) + g_\rho(x)\ t^{-2/5} \cos\theta + h_\rho$$

$$p(\vec{r},t) = p_0(x)^{-6/5} + g_p(x)\ t^{-4/5} \cos\theta + h_p$$

$$v_r(\vec{r},t) = v_0(x)^{-3/5} + g_r(x)\ t^{-1/5} \cos\theta + h_r$$

$$v_\theta(\vec{r},t) = g_\theta(x)\ t^{-1/5} \cos\theta + h_\theta$$

$$g(\vec{r},t) = R(t) + g_R(x)\ t^{-4/5}\cos\theta + h_R(t,\theta) - r \qquad (68)$$

The second order perturbed equations are thus

$$\frac{\partial h_\rho}{\partial t} + \rho\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 h_r) + \frac{1}{r}\left(\frac{\cos\theta}{\sin\theta} h_\theta + \frac{\partial h_\theta}{\partial \theta}\right)\right] + \cos^2\theta f_\rho\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{2}{r} f_\theta\right]$$

$$+ h_\rho\left[\frac{1}{r^2}\frac{\partial}{\partial \rho}(r^2 v) + v\frac{\partial h_\rho}{\partial r} + \cos^2\theta f_r \frac{\partial f_\rho}{\partial r} + h_r \frac{\partial \rho}{\partial r} - \frac{\sin^2\theta}{r} f_\theta f_\rho = 0\right.$$

$$\frac{\partial h_p}{\partial t} + \gamma p\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 h_r) + \frac{1}{r}\left(\frac{\cos\theta}{\sin\theta} h_\theta + \frac{\partial h_\theta}{\partial \theta}\right)\right] + \gamma\cos^2\theta f_p\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 f_r) + \frac{2}{r} f_\theta\right]$$

$$+ \gamma h_\rho\left[\frac{1}{r^2}\frac{\partial}{\partial \rho}(r^2 v) + v\frac{\partial h_\rho}{\partial r} + \cos^2\theta f_r \frac{\partial f_\rho}{\partial r} + h_r \frac{\partial p}{\partial r} - \frac{\sin^2\theta}{r} f_\theta f_p = 0\right.$$

$$\frac{\partial h_r}{\partial t} + v\frac{\partial h_r}{\partial r} + \cos^2\theta\, f_r \frac{\partial f_r}{\partial r} + h_r \frac{\partial v}{\partial r} - f_\theta(f_\theta + f_r)\sin^2\theta$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial r}\left(-\frac{h_\rho}{\rho} + \frac{f_\rho^2}{\rho^2}\cos^2\theta\right) + \cos^2\theta \frac{f_\rho}{\rho^2}\frac{\partial f_p}{\partial r} - \frac{1}{\rho}\frac{\partial h_p}{\partial r}$$

$$\frac{\partial h_\theta}{\partial t} + \frac{f_\theta^2}{r}\sin\theta\cos\theta + v\frac{\partial h_\theta}{\partial r} + \sin\theta\cos\theta\, f_r \frac{\partial f_\theta}{\partial r} + \frac{v h_\theta}{r} + \frac{\sin\theta\cos\theta}{r} f_r f_\theta$$

$$= -\frac{1}{r\rho}\frac{\partial h_p}{\partial \theta} - \frac{f_p}{r\rho}\frac{f_\rho}{\rho}\sin\theta\cos\theta \tag{69}$$

The second order boundary conditions are

$$h_R \frac{\partial \rho}{\partial r} + \frac{\cos\theta}{2} \frac{\partial^2 \rho}{\partial r^2} f_R^2 + \cos^2\theta \, f_R \frac{\partial f_\rho}{\partial r} + h_\rho = \frac{\gamma+1}{\gamma-1} \rho_b [- \frac{f_R}{H} \cos^2\theta + \frac{R^2}{2H^2} \cos^2\theta]$$

$$-\sin\theta \cos\theta (f_r + \frac{\partial v}{\partial r} f_R) + v \frac{\partial h_R}{\partial \theta} = -R(h_\theta + \frac{\partial f_\theta}{\partial r} f_R \cos\theta \sin\theta) - f_R f_\theta \sin\theta \cos\theta$$

$$\frac{\partial v}{\partial r} h_R + \frac{1}{2} \frac{\partial^2 v}{\partial r^2} f_R^2 \cos^2\theta + f_R \frac{\partial f_\theta}{\partial r} \cos^2\theta + h_r + \frac{\sin^2\theta}{R} f_\theta f_R$$

$$= (\frac{2p}{\gamma+1 \; \rho_b})^{1/2} \{ \frac{1}{2} [\frac{1}{p} \frac{\partial p}{\partial r} h_R + \frac{1}{2p} \frac{\partial^2 p}{\partial r^2} f_R^2 \cos^2\theta + \frac{1}{p} \frac{\partial f_p}{\partial \rho} f_R \cos^2\theta$$

$$+ \frac{\cos^2\theta}{H} (\frac{R}{p} \frac{\partial p}{\partial r} + 1) f_R + \frac{R^2}{2H^2} \cos^2\theta + \frac{\sin^2\theta}{R^2} f_R^2 + \frac{\cos^2\theta}{H} \frac{f_p}{p} R + \frac{h_p}{p}]$$

$$- \frac{\cos^2\theta}{8} ((\frac{1}{p} \frac{\partial p}{\partial r} f_R)^2 + (\frac{f_p}{p})^2 + (\frac{R}{H})^2)\}$$

$$\frac{\partial h_R}{\partial t} = \frac{\gamma+1}{2} (\frac{\partial v}{\partial r} h_R + \frac{1}{2} \frac{\partial^2 v}{\partial r^2} f_R^2 \cos^2\theta + f_R \frac{\partial f_r}{\partial r} \cos^2\theta + h_r + \frac{\sin^2\theta}{R} f_\theta f_R) \quad . \tag{70}$$

The similarity form for the solution provides a $h_r$ time dependence which allows the equations to be simplified. Furthermore, the angular dependence can be shown to allow only the following form

$$
\begin{aligned}
h_\theta &= t^{1/5}\, H_\theta \sin\theta \cos\theta \\
h_\rho &= t^{4/5}\, (H_\rho \cos^2\theta + J_\rho) \\
h_p &= t^{-2/5}\, (H_p \cos^2\theta + J_p) \\
h_r &= t^{1/5} (H_r \sin\theta \cos\theta + J_r) \\
h_R &= t^{6/5} (H_R \sin\theta \cos\theta + J_R) \quad .
\end{aligned}
\tag{71}
$$

Using both the time and angular separable dependence, the equations can finally be written in the similarity variable x as

$$\frac{4}{5} J_\rho - \frac{2}{5} xJ'_\rho + \rho[\frac{2}{ax} J_r + \frac{J'_r}{a} - \frac{1}{ax} H_\theta] + J_\gamma(\frac{2}{ax} v + \frac{v'}{a}$$

$$+ \frac{v}{a} J'_\rho + \frac{J_r}{a} \rho' - \frac{1}{ax} f_\theta f_\rho = 0$$

$$\frac{4}{5} H_\rho - \frac{2}{5} xH'_\rho + \rho[\frac{2}{ax} H_r + \frac{H'_r}{a} + \frac{3}{ax} H_\theta] + f_\rho[\frac{2}{ax} f_r + \frac{f'_r}{a} + \frac{2}{ax} f_\theta]$$

$$+ H_\rho(\frac{2}{ax} v + \frac{v'}{a}) + \frac{v}{a} H'_\rho + \frac{f_r}{a} f'_\rho + \frac{H_r}{a} \rho' + \frac{1}{ax} f_\theta f_\rho = 0$$

$$-\frac{2}{5} J_p - \frac{2}{5} xJ'_p + \gamma p[\frac{2}{ax} J_r + \frac{J'_r}{a} - \frac{1}{ax} H_\theta] + \gamma J_p(\frac{2}{ax} v + \frac{v'}{a})$$

$$+ \frac{v}{a} J'_p + \frac{J_r}{a} p' - \frac{1}{ax} f_\theta f_p = 0$$

$$-\frac{2}{5} H_p - \frac{2}{5} xH'_p + \gamma p[\frac{2}{ax} H_r + \frac{H'_r}{a} + \frac{3}{ax} H_\theta] + \gamma f_p(\frac{2}{ax} f_r + \frac{f'_r}{a} + \frac{2}{ax} f_\theta]$$

$$+ H_p(\frac{2}{ax} v + \frac{v'}{a}) + \frac{v}{a} H'_p + \frac{f_r}{a} f'_p + \frac{H_r}{a} p' + \frac{f_\theta f_p}{ax} = 0$$

$$\frac{1}{5} H_\theta - \frac{2}{5} xH'_\theta + \frac{f_\theta^2}{ax} + \frac{v}{a} H'_\theta + \frac{f_r}{a} f'_\theta + \frac{vH_\theta}{ax} + \frac{f_r f_\theta}{ax} = \frac{2}{ax\rho} H_p - \frac{f_p f_\rho}{ax\rho^2}$$

$$\frac{1}{5} J_r - \frac{2}{5} xJ'_r + \frac{v}{a} J'_r + \frac{J_r}{a} v' - \frac{f_\theta(f_\theta + f_r)}{ax} = \frac{1}{a\rho^2} J_\rho p' - \frac{J'_p}{a\rho}$$

$$\frac{1}{5} H_r - \frac{2}{5} xH'_r + \frac{v}{a} H'_r + \frac{f_r}{a} f'_r + \frac{H_r}{a} v' + \frac{f_\theta}{ax} (f_\theta + f_r) =$$

$$-\frac{1}{\rho} \frac{p'}{a}(-\frac{H_\rho}{\rho} + \frac{f_\rho^2}{\rho^2}) + \frac{f_\rho}{a\rho^2} f'_p - \frac{1}{a\rho} H'_p \qquad (72)$$

Where the boundary conditions are

$$\frac{J_R}{a}\rho' + J_\rho = 0$$

$$\frac{H_R}{a}\rho' + \frac{f_R^2}{2a^2}\rho'' + \frac{f_R}{a} f_\rho + H_\rho = \frac{\gamma+1}{\gamma-1}\rho_b\left(-\frac{f_R}{H} + \frac{a^2}{2H^2}\right)$$

$$f_R\left(f_r + \frac{f_R}{a} f_r'\right) - 2vH_R = -a\left(H_\theta + \frac{f_R}{a} f_\theta'\right) - f_R f_\theta$$

$$\frac{J_R}{a} v' + J_r + \frac{f_\theta f_R}{a} = \left(\frac{2p}{(\gamma+1)\rho_b}\right)^{1/2} \left\{ \frac{1}{2}\left[\frac{p'}{ap} J_R + \frac{f_R^2}{a^2} + \frac{J_p}{p}\right]\right\}$$

$$\frac{H_R}{a} v' + \frac{f_R^2}{2a^2} v'' + \frac{f_R}{a} f_r' + H_r - \frac{f_\theta f_R}{a}$$

$$\left(\frac{2p}{(\gamma+1)\rho_0}\right)^{1/2}\left\{ \frac{1}{2}\left[ \frac{p'}{ap} H_R + \frac{f_R^2}{2p}\frac{p''}{a^2} + \frac{f_R}{p}\frac{f_p}{a} + \frac{f_R}{H}\left(\frac{a}{p}\frac{p'}{a} + 1\right)\right.\right.$$

$$\left. + \frac{a^2}{2H} - \frac{f_R^2}{a^2} + \frac{f_p}{Hp} a + \frac{H_p}{p}\right]$$

$$\left. - \frac{1}{8}\left[\left(\frac{p'}{ap} f_R\right)^2 + \left(\frac{f_p}{p}\right)^2 + \left(\frac{a}{H}\right)^2\right]\right\}$$

$$\frac{6}{5} J_R - \frac{2}{5} J'_R = \frac{\gamma+1}{2}\left(J_R \frac{v'}{a} + J_r + \frac{f_\theta f_R}{a}\right)$$

$$\frac{6}{5} H_R - \frac{2}{5} H'_R = \frac{\gamma+1}{2}\left(H_R \frac{v'}{a} + \frac{f_R^2}{2a^2} v'' + f_R \frac{f_r'}{a} + H_r + \frac{f_\theta f_R}{a}\right)$$

all evaluated at x=1. Prime indicates derivative with respect to x.

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